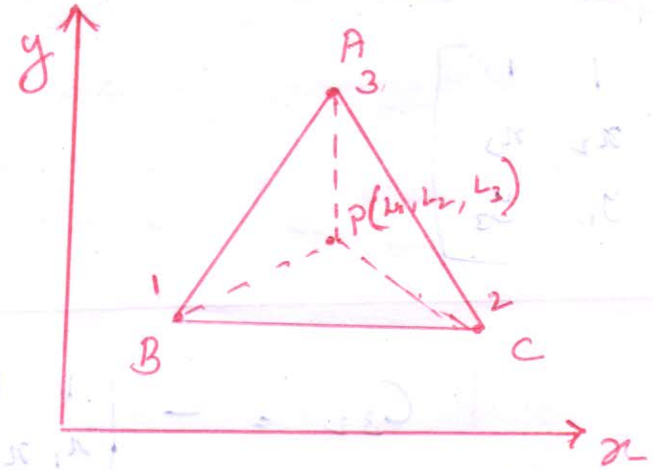
MODULE. 3

Natural Co-ordinates in 2D..



Let's. Consider a triangular element having 3 nodes as shown in figure. P be a point inside the element at it has 3. coordn L_1, L_2 & L_3

From the definition of natural co-ordinates, we know that

$$\left.\begin{aligned} L_1 + L_2 + L_3 &= 1 \\ L_1 x_1 + L_2 x_2 + L_3 x_3 &= x \\ L_1 y_1 + L_2 y_2 + L_3 y_3 &= y \end{aligned}\right\} \rightarrow (1)$$

representing in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

$$D^{-1} = \frac{C^T}{|D|}$$

$$C_{11} = +\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} = x_2y_3 - x_3y_2$$

$$C_{12} = -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} = x_3y_1 - x_1y_3$$

$$C_{13} = +\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ y_2 & y_1 \end{vmatrix} = y_2 - y_1$$

$$C_{22} = +\begin{vmatrix} 1 & 1 \\ y_1 & y_3 \end{vmatrix} = y_3 - y_1$$

$$C_{23} = -\begin{vmatrix} 1 & 1 \\ y_1 & y_2 \end{vmatrix} = y_1 - y_2$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ x_2 & x_3 \end{vmatrix} = x_3 - x_2$$

$$C_{32} = -\begin{vmatrix} 1 & 1 \\ x_1 & x_3 \end{vmatrix} = x_1 - x_3$$

$$C_{33} = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$$

$$|D| = (x_2y_3 - x_3y_2) - 1\,(x_1y_3 - x_3y_1) + 1\,(x_1y_2 - x_2y_1)$$

$$C = \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

We need C^T

$$D^{-1} = \frac{C^T}{|D|}$$

$$= \frac{1}{(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \quad " \quad " \quad \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \rightarrow ②$$

the area of triangle ABC can be given by.

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$A = \frac{1}{2} \left[x_2y_3 - x_3y_2 - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1) \right]$$

$$2A = (x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1) \rightarrow ③$$

Sub: 3 in ② we get.

$$\boxed{\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}}$$

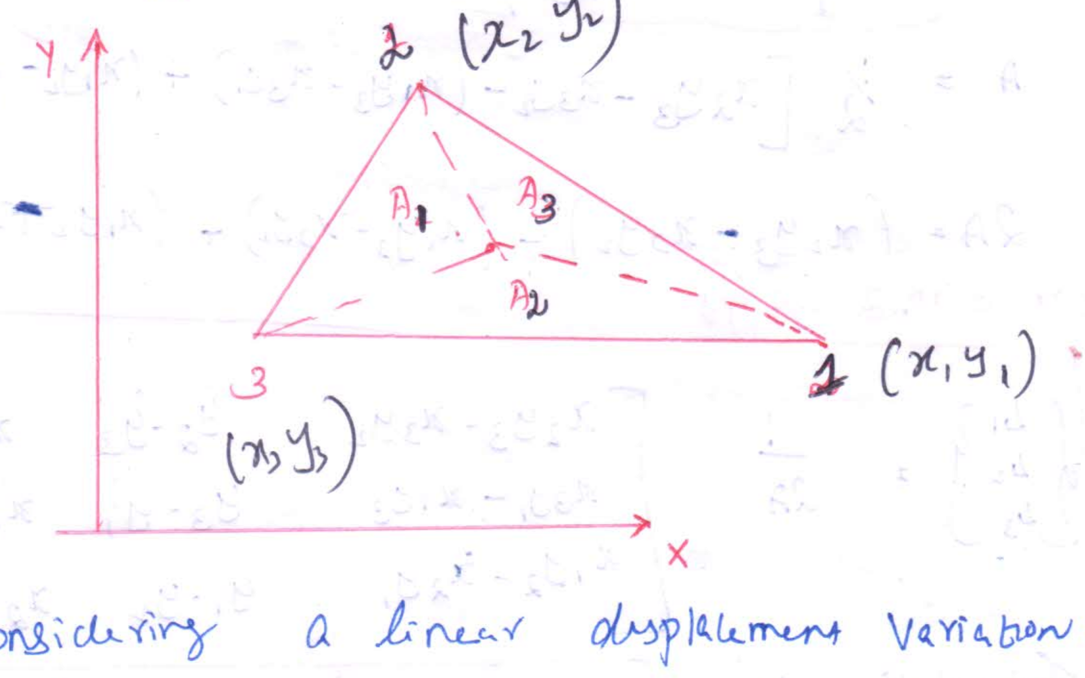
Integration of polynomials term in natural co-ordn. for 2D elements can

$$\oint_A (L_1)^\alpha (L_2)^\beta (L_3)^\gamma \, dA = \frac{\alpha!\, \beta!\, \gamma!}{(\alpha+\beta+\gamma+2)!} \times 2A.$$

Shape functions using Area Co-ordinates.

The interpolation functions for a triangular element are algebraically complex if expressed in Cartesian Co-ordinates, moreover, the integration required to obtain the element stiffness matrix is complex.

The interpolation function & subsequently the required integration can be obtained in a simplified manner by the Concept of area coordinates.



Considering a linear displacement variation of a triangular element is shown in above fig. the displacement at any point can be written as.

$$u = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3.$$

(or)

$$u = \{\phi\}^T \{\alpha\} \rightarrow (1)$$

$$L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A}.$$

$A \rightarrow$ area of triangle 1, 2, 3.

$A_1 + A_2 + A_3 = A.$

$L_1 + L_2 + L_3 = 1.$

$$L_1 = \frac{A_1}{A} = \frac{2A_1}{2A} = \frac{\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}}$$

$$L_1 = \frac{1}{2A}\left[(x_2 y_3 - y_2 x_3) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$2A = \left[(x_2 y_3 - y_2 x_3) + (x_3 y_1 - y_3 x_1) + (x_1 y_2 - y_1 x_2)\right]$$

L_2 & L_3 can be obtained similarly.

$$L_2 = \frac{1}{2A} \begin{vmatrix} 1 & x & y \\ 1 & x_3 & y_3 \\ 1 & x_1 & y_1 \end{vmatrix}$$

$$= \frac{1}{2A}\left[(x_3 y_1 - y_3 x_1) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$L_3 = \frac{1}{2A} \begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix}$$

$$= \frac{1}{2A}\left[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

In gener the area Co-ordinates can be written as

$$L_i = a_i + b_i x + c_i y$$

$$a_i = (x_j y_k - y_j x_k)/2A.$$

$$b_i = (y_j - y_k)/2A$$

$$c_i = (x_k - x_j)/2A.$$

for $i=1, j=2, k=3$

$i=2, j=3, k=1$

$i=3, j=1$ & $k=2$.

at node 1, $L_1 = 1$, $L_2 = L_3 = 0$.

$L_2 = 1$, $L_1 = L_3 = 0$

$L_3 = 1$, $L_1 = L_2 = 0$.