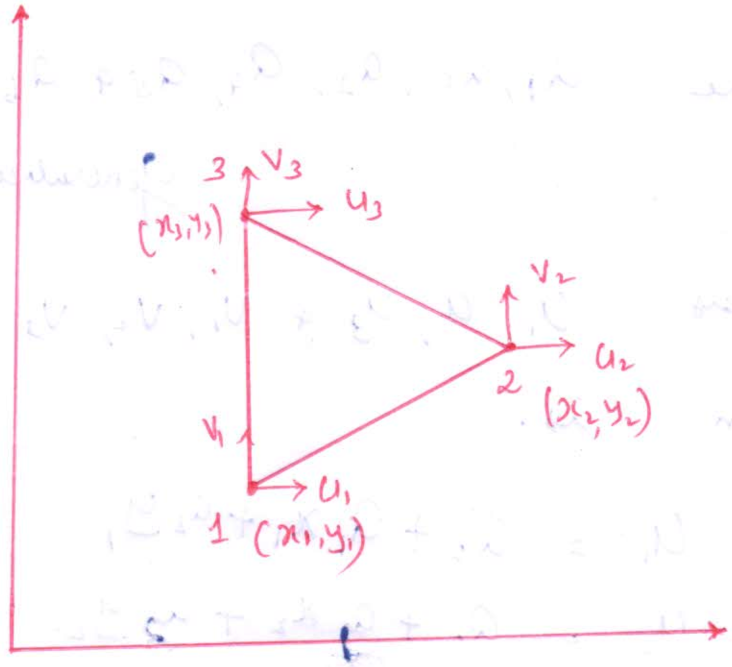
Shape Function Derivation for the triangle element (Shape func for CST)



Lets consider a triangle elem (CST), with nodes 1, 2, & 3 as shown in the figure

$$\text{Displacement } \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ V_1 \\ V_2 \\ V_3 \end{Bmatrix} \quad \{U, V\} \longrightarrow (1)$$

Since the (CST) elements have 2 DOF in each node (u, v), the total dof is 6. So it has six generalized co-ordinates.

Let assume by interpolation of polynomial.

$$\left.\begin{aligned} u &= a_1 + a_2 x + a_3 y \\ v &= a_4 + a_5 x + a_6 y \end{aligned}\right\} \rightarrow (2)$$

Where a_1, a_2, a_3, a_4, a_5 & a_6 are global or generalized Co-ordinates.

So these u_1, u_2, u_3 & v_1, v_2, v_3 Can be written as.

$$\left.\begin{aligned} u_1 &= a_0 + a_1 x_1 + a_2 y_1 \\ u_2 &= a_0 + a_2 x_2 + a_3 y_2 \\ u_3 &= a_0 + a_1 x_3 + a_3 y_3 \end{aligned}\right\} \rightarrow (3)$$

my

$$v_1 = a_4 + a_5 x_1 + a_6 y_1$$
$$v_2 = a_4 + a_5 x_2 + a_6 y_2$$
$$v_3 = a_4 + a_5 x_3 + a_6 y_3$$

Consider the displacement in x plane & represent in matrix form.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} \rightarrow (4)$$

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}_{D}^{-1} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \rightarrow 5$$

$$D^{-1} = \frac{C^T}{|D|}$$

$$C_{11} = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_3)$$

$$C_{12} = -\begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} = y_2 - y_3$$

$$C_{13} = \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = (x_3 - x_2)$$

$$C_{21} = -\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} = x_3 y_1 - x_1 y_3$$

$$C_{22} = \begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} = y_3 - y_1$$

$$C_{23} = -\begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} = x_1 - x_3$$

$$C_{31} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = (x_1 y_2 - x_2 y_1)$$

$$C_{32} = -\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = y_1 - y_2$$

$$C_{33} = \begin{vmatrix} 1 & x_1 \\ 1 & y_2 \end{vmatrix} = x_2 - x_1$$

$$C = \begin{bmatrix} (x_2y_3 - x_3y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3y_1 - x_1y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1y_2 - y_1x_2) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix}$$

$$C^T = \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}$$

$$D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$|D| = (x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2) \Rightarrow (i)$$

$$A = \frac{1}{2}\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$|A| = \frac{1}{2}\left[(x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2)\right]$$

$$2A = (x_2y_3 - x_3y_2) - x_1(y_3 - y_2) + y_1(x_3 - x_2) \quad (ii)$$

Compare (i) & (ii) and

subs in (5) we get.

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

13

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$p_1 = x_2y_3 - x_3y_2 \quad p_2 = x_3y_1 - x_1y_3 \quad p_3 = x_1y_2 - x_2y_1$

$q_1 = y_2 - y_3 \quad q_2 = y_3 - y_1 \quad q_3 = y_1 - y_2$

$r_1 = x_3 - x_2 \quad r_2 = x_1 - x_3 \quad r_3 = x_2 - x_1$

$$= \frac{1}{2A} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \rightarrow (6)$$

We know that from eqn (1)

$$u = a_1 + a_2x + a_3y$$

$$u = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \rightarrow (7)$$

sub (6) in (7) we get

$$u = \begin{bmatrix} 1 & x & y \end{bmatrix} \times \frac{1}{2A} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} p_1 + q_1x + r_1y, & p_2 + q_2x + r_2y, & p_3 + q_3x + r_3y \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{p_1 + q_1x + r_1y}{2A}, & \frac{p_2 + q_2x + r_2y}{2A}, & \frac{p_3 + q_3x + r_3y}{2A} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1, & N_2, & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}.$$

$$N_1 = \frac{P_1 + q_1 x + r_1 y}{2A}, \quad N_2 = \frac{P_2 + q_2 x + r_2 y}{2A}$$

$$N_3 = P_3 + q_3 x + r_3 y.$$

$$u = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

III^y

$$V = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

Displacement function $u = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$

or

$$= \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$