Potential Energy Approach.

The general expression for potential energy is given by.

$$PE = (U - W_{ext})$$

The strain energy density function in the negative of the work done (per unit vol) by the internal forces in an elastic body through the displacements u_i. Thus because the system of internal force is conserve, it is equal to the work these forces would perform in returning to the unstrained position.

$$\pi = \frac{1}{2}\int_l \sigma^T e \cdot A \cdot dx - \int_l u^T f A \cdot dx - \int_l u^T \cdot T \cdot dx - \int_l \sum_i u_i P_i$$

$\sigma \to$ stress, $e \to$ strain, $A \to$ Area, $u \to$ Displacement.
$f \to$ body force, $T \to$ traction force, $P \to$ point load.

When the continuum has been discretized into finite elements, the expression for π becomes.

$$\pi = \sum_e \frac{1}{2}\int_e \sigma^T e A dx - \sum_e \int_e u^T \cdot T \cdot dx - \sum_e \int u^T f A \cdot dx - \sum_i Q_i P_i$$

the above equation can be written as

$$\pi = \sum_e U_e - \sum_e \int_e u^T f A \cdot dx - \sum_e \int_e u^T \cdot T \cdot dx - \sum Q_i P_i$$

$\boxed{\text{Strain energy } U_e = \frac{1}{2}\int \sigma^T e \, A \cdot dx}$

Stiffness matrix for a bar element:

$$U_e = \frac{1}{2}\int \sigma^T e \, A \cdot dx$$

$e = Bu.$
$\sigma = Ee = EBu.$

$$= \frac{1}{2}\int (EBu)^T (Bu) \cdot A \cdot dx.$$

$$= \frac{1}{2}\int E B^T u^T B u A \cdot dx = \frac{1}{2} u^T \left[A E B^T B u \int_0^l dx \right]$$

$$= \frac{1}{2} u^T \left[A E B^T B u \cdot l \right]$$

$$[B] = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix}, \quad B^T = \begin{Bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{Bmatrix}$$

$$u_e = \frac{1}{2} u^T A E \begin{Bmatrix} -\frac{1}{l} \\ \frac{1}{l} \end{Bmatrix} \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \cdot u l$$

$$= \frac{1}{2} u^T A E \times \frac{1}{l} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \times \frac{1}{l} \begin{bmatrix} -1 & 1 \end{bmatrix} u \cdot l$$

$$= \frac{1}{2} u^T A E \times \frac{1}{l^{\cancel{2}}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot u \times \cancel{l}$$

$$= \frac{1}{2} u^T \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u$$

the above eqn is of the form.

$$u_e = \frac{1}{2} u^T \cdot [K] \cdot u.$$

$$\therefore [K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

work done = Strain energy (P)

$$= \frac{1}{2} \{W\}^T \{\omega^*\}$$

$$= \frac{1}{2} \{[K]\{\omega^*\}\}^T \{\omega^*\}$$

$$= \frac{1}{2} \{\omega^*\}^T [K] \{\omega^*\}$$