Coordinates

1. Global Co-ordinates

   The point in the entire structure are defined using co-ordinate system is known as global coordinate.

2. Local Co-ordinates

   In fem separate coordinate is used for each element termed.

3. Natural Co-ordinates

   used to define any point inside the element by a set of dimensionless numbers whose magnitude never exceeds unity.

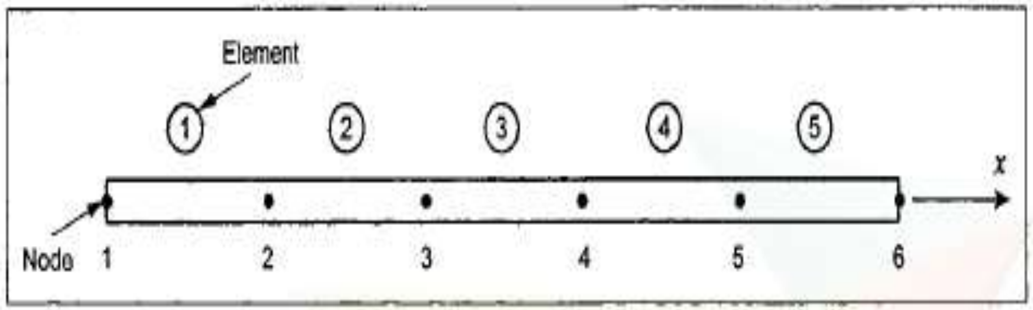


Fig. 2.7. One dimensional bar

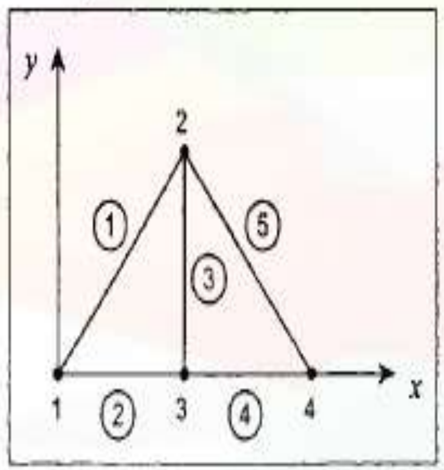
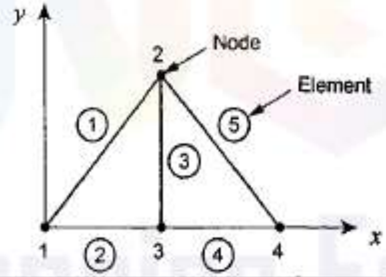
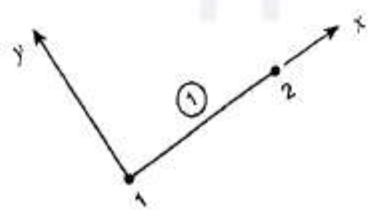


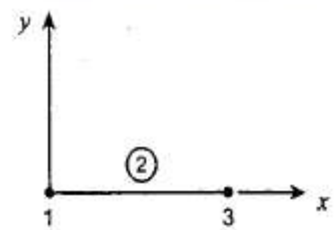
Fig. 2.8. Two dimensional triangular element



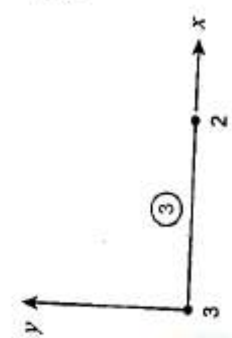
For element (1):



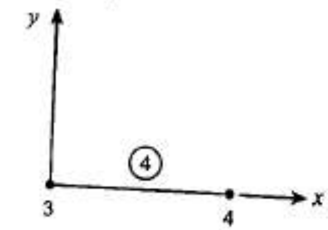
For element (2):



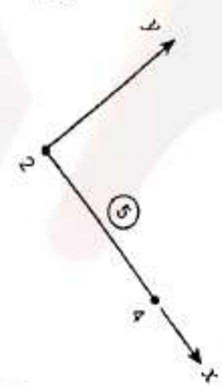
3):



For element (4):



For element (5):



(1) Natural Co-ordinates in One Dimension

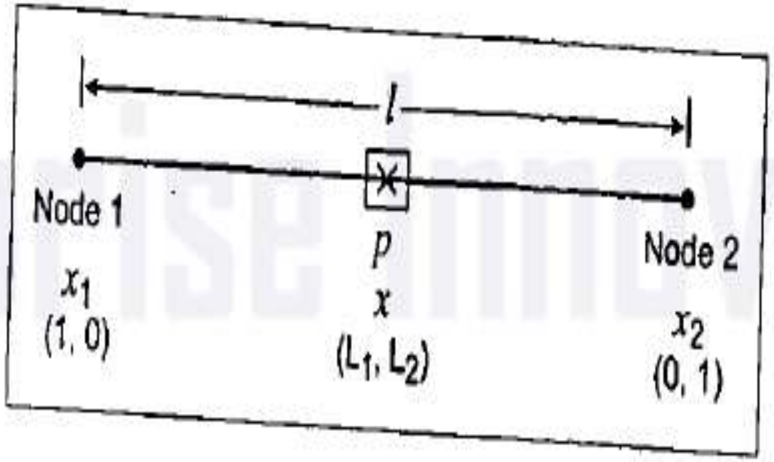


Fig. 2.10. Natural co-ordinates for a line element

Consider a two noded line element as shown in Fig.2.10. Any point p inside the line element is identified by two natural co-ordinates L_1 and L_2 and the cartesian co-ordinate x. Node 1 and node 2 have the cartesian co-ordinates x_1 and x_2 respectively.

We know that,

Total weightage of natural co-ordinates at any point is unity.

i.e., $$L_1 + L_2 = 1 \quad \text{... (2.1)}$$

Any point x within the element can be expressed as a linear combination of the nodal co-ordinates of nodes 1 and 2 as,

$$L_1 x_1 + L_2 x_2 = x \quad \text{... (2.2)}$$

Arrange equation (2.1) and (2.2) in matrix form,

$$\begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$= \frac{1}{(x_2 - x_1)} \begin{bmatrix} x_2 & -1 \\ -x_1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \end{Bmatrix}$$

$$\left[\textit{Note:} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{(a_{11} \cdot a_{22}) - (a_{12} \cdot a_{21})} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right]$$

$$= \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 - x \\ -x_1 + x \end{Bmatrix}$$

$$= \frac{1}{x_2 - x_1} \begin{Bmatrix} x_2 - x \\ x - x_1 \end{Bmatrix}$$

$$= \frac{1}{l} \begin{Bmatrix} x_2 - x \\ x - x_1 \end{Bmatrix}$$ [$\because x_2 - x_1$ is the length of the element, l]

$$\begin{Bmatrix} L_1 \\ L_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{x_2 - x}{l} \\ \dfrac{x - x_1}{l} \end{Bmatrix}$$

The variation of L_1 and L_2 is shown in Fig.2.12 and Fig.2.13. L_1 is one at node 1 and it is zero at node 2 whereas L_2 is one at node 2 and it is zero at node 1.

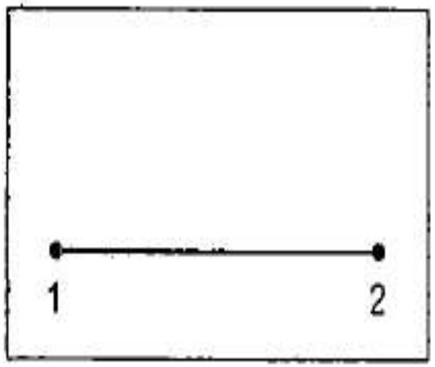


Fig. 2.11.

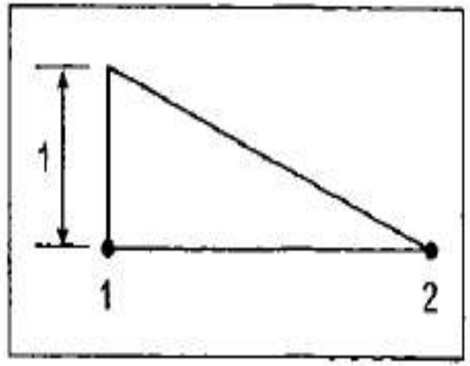


Fig. 2.12. Variation of L_1

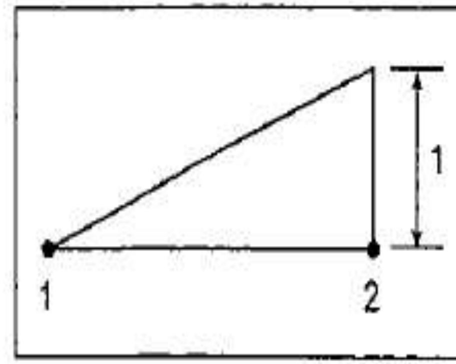


Fig. 2.13. Variation of L_2

Integration of polynomial terms in natural co-ordinates can be performed by using the simple formula,

$$\int_{x_1}^{x_2} (L_1)^{\alpha} (L_2)^{\beta} \, dx = \frac{\alpha! \, \beta!}{(\alpha + \beta + 1)!} \times l_x \quad \ldots (2.3)$$

where, $\alpha!$ is the factorial of α.

Natural Co-ordinate, ε

In one dimensional problem, the following type of natural co-ordinate is also used.

Consider a one dimensional element as shown in Fig.2.14.

In the local number scheme, the first node will be numbered 1 and the second node 2. c is the centre of nodes 1 and 2 and p is the point referred.

The natural co-ordinator ε for any point in the element is defined as,

$$\varepsilon = \frac{pc}{\left(\frac{x_2 - x_1}{2}\right)}$$

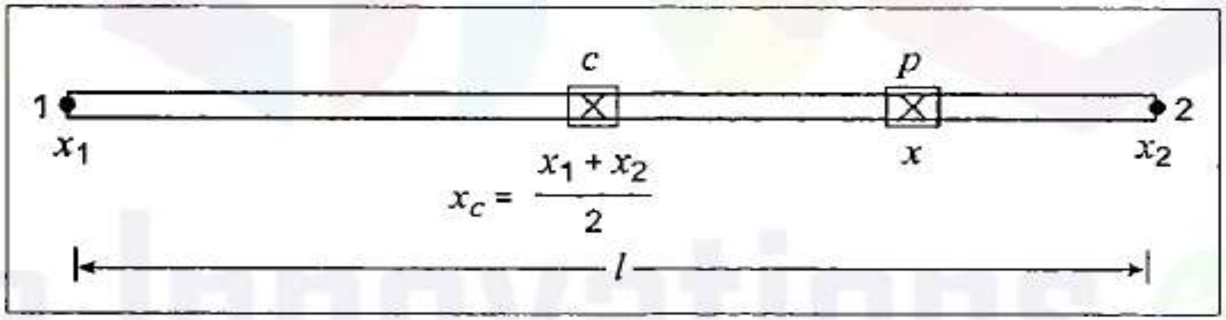


Fig. 2.14.

$$\Rightarrow \qquad \varepsilon = \frac{pc}{\frac{l}{2}} \qquad [\because x_2 - x_1 = l]$$

$$= \frac{2}{l} pc = \frac{2}{l}(x - x_c) \qquad [\because pc = x - x_c]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{x_1 + x_2}{2} \right) \right] \qquad \left[\because \; x_c = \frac{x_1 + x_2}{2} \right]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{x_2 + x_1}{2} \right) \right]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{x_2 - x_1 + 2x_1}{2} \right) \right]$$

$$= \frac{2}{l} \times \left[x - \left(\frac{l + 2x_1}{2} \right) \right]$$

$$\varepsilon = \frac{2}{l} \left[x - \left(\frac{l}{2} + x_1 \right) \right]$$

$$\Rightarrow \quad \frac{\varepsilon l}{2} = x - \frac{l}{2} - x_1$$

$$\Rightarrow \quad \frac{\varepsilon l}{2} + \frac{l}{2} = x - x_1$$

$$\Rightarrow \quad \boxed{\frac{l}{2}(\varepsilon + 1) = x - x_1} \qquad \ldots (2.4)$$

Applying boundary conditions.

Applying boundary conditions,

At node 1, $x = x_1$

(2.4) $\Rightarrow$ $\frac{l}{2}(1+\varepsilon) = 0$

$\Rightarrow$ $1+\varepsilon = 0$

$\Rightarrow$ $\boxed{\varepsilon = -1}$

At node 2, $x = x_2$

(2.4) $\Rightarrow$ $\frac{l}{2}(1+\varepsilon) = x_2 - x_1$

$\frac{l}{2}(1+\varepsilon) = l$

$\Rightarrow$ $1+\varepsilon = 2$

$\Rightarrow$ $\boxed{\varepsilon = 1}$

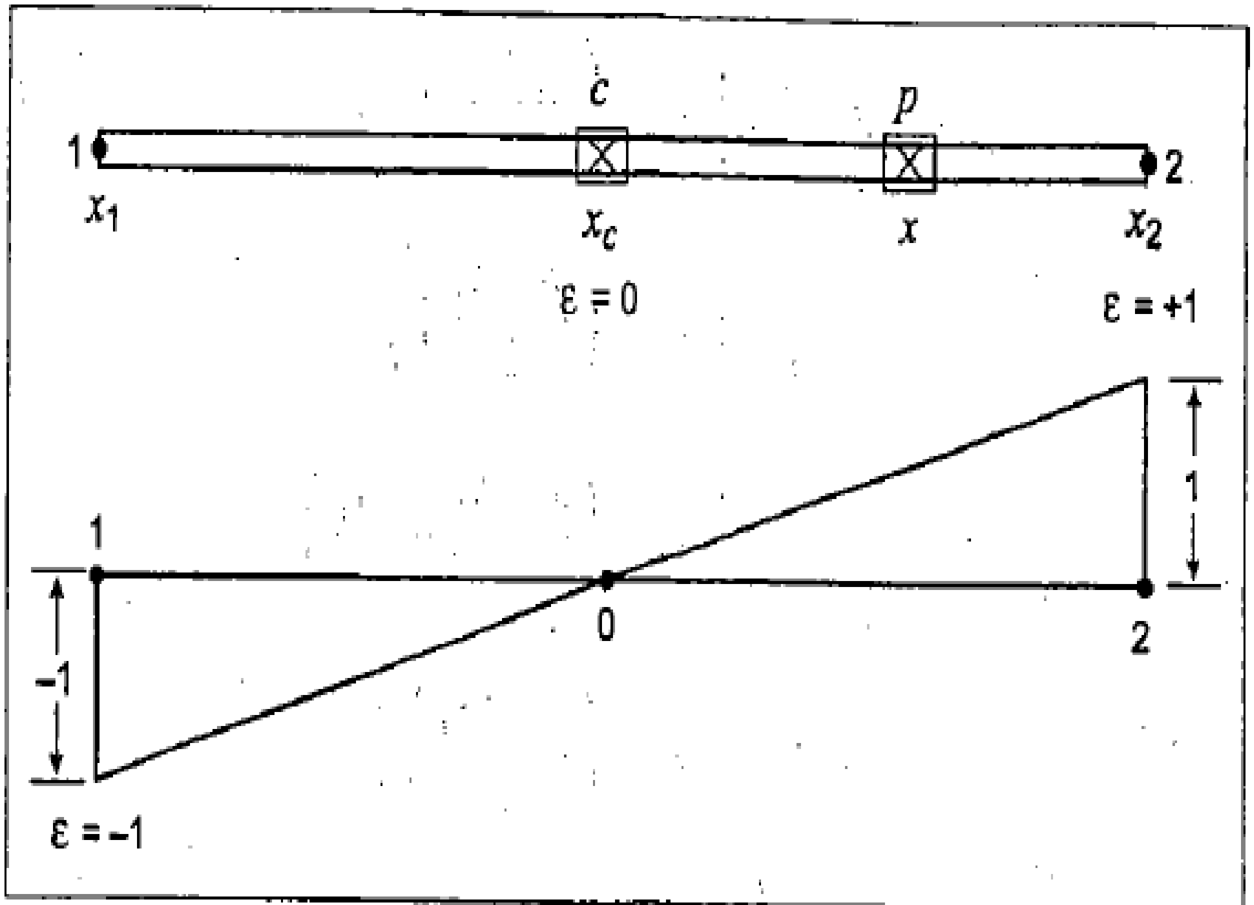


Fig. 2.15. Variation of natural co-ordinate, ε

Shape Functions

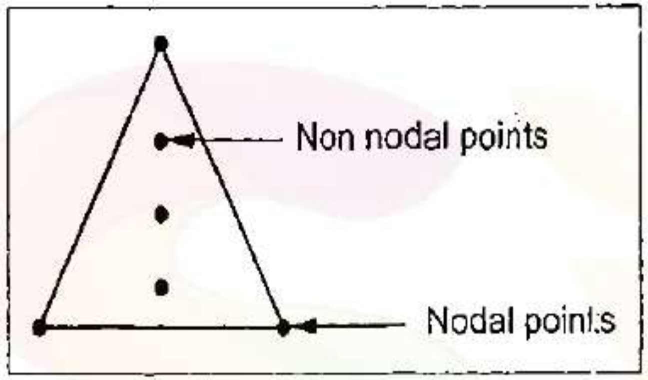


Fig. 2.17.

Consider the three noded triangular element as shown in Fig.2.17.

The nodes are exterior and at any point within the element the field variable is described by the following approximate relation.

$$\phi(x,y) = N_1(x,y)\,\phi_1 + N_2(x,y)\,\phi_2 + N_3(x,y)\,\phi_3$$

where ϕ_1, ϕ_2, ϕ_3 are the values of the field variable at the nodes, and N_1, N_2 and N_3 are the interpolation functions. N_1, N_2 and N_3 are also called as shape functions because they are used to express the geometry or shape of the element. Shape function has unit value at one nodal point and zero value at other nodal points.

In one dimensional problem, the basic field variable is displacement.

So, $$u = \Sigma N_1 u_i \quad \text{where } u \rightarrow \text{Displacement.}$$

For two noded bar element, the displacement at any point within the element is given by,

$$u = \Sigma N_i u_i = N_1 u_1 + N_2 u_2$$

where, u_1 and u_2 are nodal displacements.

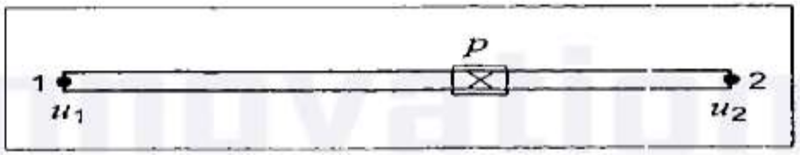


Fig. 2.18.

In two dimensional stress analysis problem, the basic field variable is displacement.

So, $u = \Sigma N_i u_i$

$v = \Sigma N_i v_i$

For three noded triangular element, the displacement at any point within the element is given by,

$$u = \Sigma N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = \Sigma N_i v_i = N_1 v_1 + N_2 v_2 + N_3 v_3$$

where, u_1, u_2, u_3, v_1, v_2 and v_3 are nodal displacements.

In general, shape functions need to satisfy the following:

1. First derivatives should be finite within an element.
2. Displacement should be continuous across the element boundary.

The characteristics of shape function are:

1. The shape function has unit value at its own nodal point and zero value at other nodal points.
2. The sum of shape function is equal to one.
3. The shape functions for two dimensional elements are zero along each side that the node does not touch.
4. The shape functions are always polynomials of the same type as the original interpolation equations.

2.6.3. Derivation of the displacement function u and shape function N for one dimensional Linear bar element based on global co-ordinate approach

Consider a bar element with nodes 1 and 2 as shown in Fig.2.20. u_1 and u_2 are the displacements at the respective nodes. So, u_1 and u_2 are considered as degrees of freedom of this bar element.

[*Note:* Degrees of freedom is nothing but nodal displacements.]

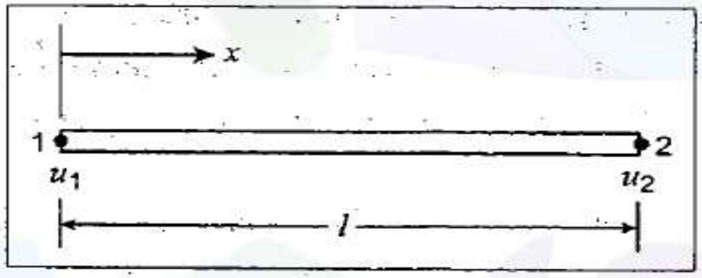


Fig. 2.20. Two noded bar element

Since the element has got two degrees of freedom, it will have two generalized co-ordinates.

$$\Rightarrow \qquad u = a_0 + a_1 x \qquad \text{... (2.15)}$$

where, a_0 and a_1 are global or generalized co-ordinates.

Writing the equation (2.15) in matrix form,

$$u = [1 \quad x] \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \qquad \text{... (2.16)}$$

At node 1, $\quad u = u_1 \, ; \; x = 0$

At node 2, $\quad u = u_2 \, , \; x = l$

Substitute the above values in equation (2.15),

$$u_1 = a_0 \qquad \text{... (2.17)}$$

$$u_2 = a_0 + a_1 l \qquad \text{... (2.18)}$$

Arranging the equation (2.17) and (2.18) in matrix form,

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} \quad \ldots (2.19)$$

$$\begin{matrix} \downarrow & & \downarrow & \downarrow \\ u^* & & C & A \end{matrix}$$

where, u^* → Degrees of freedom.

C → Connectivity matrix.

A → Generalised or global co-ordinates matrix.

$$\Rightarrow \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & l \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l-0} \begin{bmatrix} l & -0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\left[\textit{Note:} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{(a_{11}a_{22} - a_{12}a_{21})} \times \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \right]$$

$$\Rightarrow \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Substitute $\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$ values in equation (2.16),

$$\Rightarrow \; u = [1 \;\; x] \, \frac{1}{l} \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} [1 \;\; x] \begin{bmatrix} l & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{l} [l - x \quad 0 + x] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

[$\because$ Matrix multiplication $(1 \times 2) \times (2 \times 2) = (1 \times 2)$]

$$u = \left[\frac{l - x}{l} \;\; \frac{x}{l} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \ldots (2.20)$$

$$u = [N_1 \;\; N_2] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Displacement function, $u = N_1 u_1 + N_2 u_2$ ... (2.21)

where, Shape function, $N_1 = \frac{l - x}{l}$; Shape function, $N_2 = \frac{x}{l}$

We may note that N_1 and N_2 obey the definition of shape function, *i.e.*, the shape function will have a value equal to unity at the node to which it belongs and zero value at other nodes.

Checking: At node 1, $x = 0$.

$$\Rightarrow N_1 = \frac{l - x}{l} = \frac{l - 0}{l}$$

$$\boxed{N_1 = 1}$$

$$\Rightarrow N_2 = \frac{x}{l} = \frac{0}{l}$$

$$\boxed{N_2 = 0}$$

At node 2, $x = l$

$$\Rightarrow N_1 = \frac{l - x}{l} = \frac{l - l}{l}$$

$$\boxed{N_1 = 0}$$

$$\Rightarrow N_2 = \frac{x}{l} = \frac{l}{l}$$

$$\boxed{N_2 = 1}$$