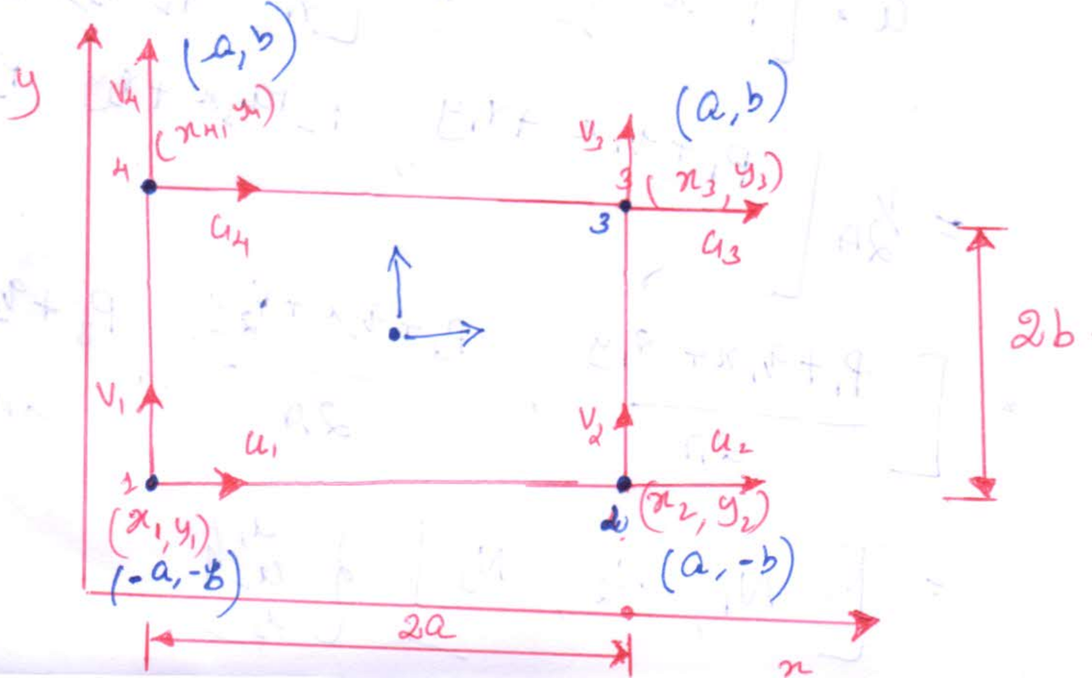
Rectangular Element (Shape function)



Interpolation fn

$$u = a_0 + a_1 x + a_2 y + a_3 xy$$
$$v = a_4 + a_5 x + a_6 y + a_7 xy \quad \rightarrow (1)$$

$$u = \begin{bmatrix} 1 & x & y & xy \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} \rightarrow (2)$$

for diffn values of u_1, u_2, u_3, u_4.

$$\left.\begin{aligned} u_1 &= a_0 + a_1 x_1 + a_2 y_1 + a_3 x_1 y_1 \\ u_2 &= a_0 + a_1 x_2 + a_2 y_2 + a_3 x_2 y_2 \\ u_3 &= a_0 + a_1 x_3 + a_2 y_3 + a_3 x_3 y_3 \\ u_4 &= a_0 + a_1 x_4 + a_2 y_4 + a_3 x_4 y_4 \end{aligned}\right]$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \\ 1 & x_4 & y_4 & x_4 y_4 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \rightarrow (3)$$

sub (3) in (1) we get :

$$N_1 = \frac{1}{4ab}(x-a)(y-b)$$

$$N_2 = \frac{1}{4ab}(x+a)(y-b)$$

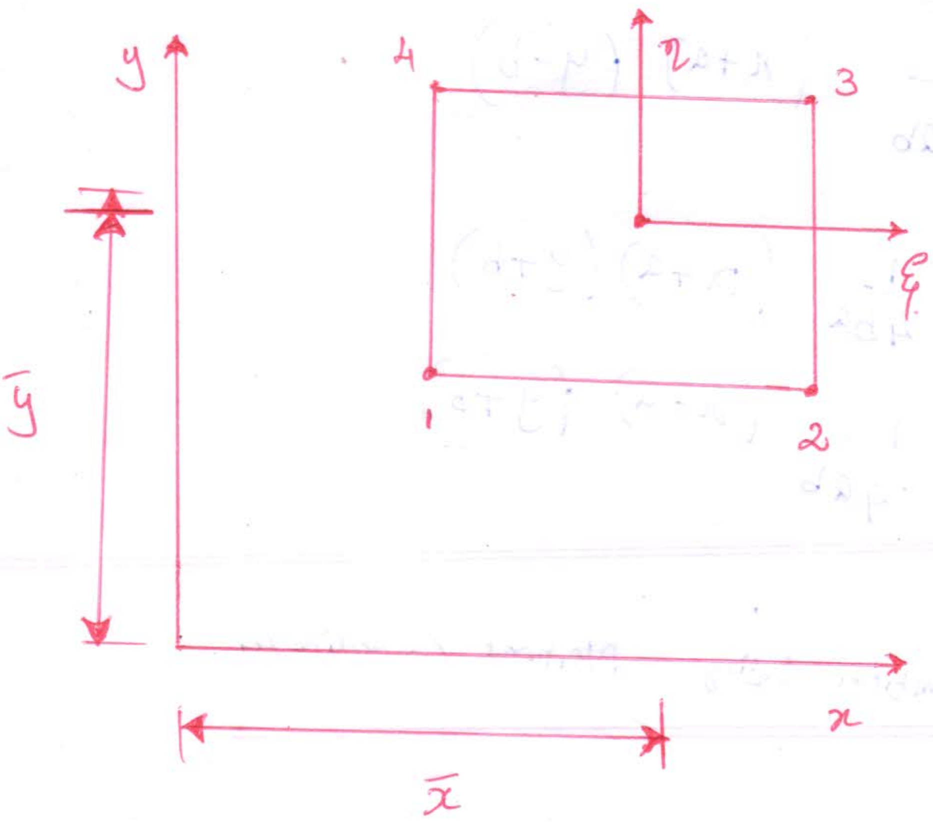
$$N_3 = \frac{1}{4ba}(x+a)(y+b)$$

$$N_4 = \frac{1}{4ab}(x-a)(y+b)$$

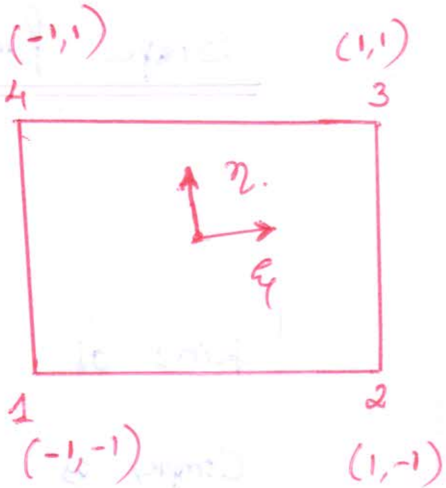
Shape function using Natural Co-ordinates:

The derivation of interpolation function in terms of Cartesian Co-ordinate System is algebraically Complex as seen in previous types session. This Complexity Can be reduced by the use of natural Co-ordinate System. The natural Co-ordinate will vary from -1 to +1 in place of -a to +a or -b to +b.

The transformation of Cartesian Co-ordinates to Natural Co-ordinates are Shown.



(a) Transformation of Cartesian to Natural Co-ordinate.



Natural co-ordinates of nodes.

From the dgm,

$$\xi = \frac{x-\bar{x}}{a} \quad \& \quad \eta = \frac{y-\bar{y}}{b} \rightarrow (1)$$

(2a & 2b are width & height)

here, the co-ordinate of the center of the rectangle can be given

by $\bar{x} = \dfrac{x_1 + x_2}{2}$, $\bar{y} = \dfrac{y_1 + y_4}{2}$

With the above relations variations of ξ & η will be from -1 to 1. Now the interpolation function can be derived similar to previous case.

The filled variable can be written in natural co-ordinate system ensuring inter-element continuity as

$$u(\xi,\eta) = a_0 + a_1\xi + a_2\eta + a_3\xi\eta \rightarrow (2)$$

$$u_1 = a_0 + a_1\xi_1 + a_2\eta_1 + a_3\xi_1\eta_1$$

$$u_2 = a_0 + a_1\xi_2 + a_2\eta_2 + a_3\xi_2\eta_2$$

$$u_3 = a_0 + a_1\xi_3 + a_3\eta_3 + a_3\xi_3\eta_3$$

$$u_4 = a_0 + a_1\xi_4 + a_4\eta_4 + a_3\xi_3\eta_3.$$

apply the value of ξ & η @ 1, 2, 3, 4. nodes and represent in matrix form.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \rightarrow (3)$$

$(2) \Rightarrow$

$$u = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \rightarrow (4)$$

~~Substn (3) in (4) we get~~

$$\begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \rightarrow (5)$$

Substn (5) in (4) we get

$$\{u\} = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$= \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$N_1 = \frac{(1-\xi)(1-\eta)}{4} \qquad N_3 = \frac{(1+\xi)(1+\eta)}{4}$$

$$N_2 = \frac{(1+\xi)(1-\eta)}{4} \qquad N_4 = \frac{(1-\xi)(1+\eta)}{4}$$