Strain Displacement Matrix [B] for CST ELEMENT.

We know

$$\boxed{\begin{aligned} u &= N_1 u_1 + N_2 u_2 + N_3 u_3 \\ v &= N_1 v_1 + N_2 v_2 + N_3 v_3. \end{aligned}}$$

The strain components for CST elements are

Normal strain $\boxed{e_x = \frac{\partial u}{\partial x}, \; e_y = \frac{\partial v}{\partial y}}$, Shear strain $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

$$e_x = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3$$

$$e_y = \frac{\partial N_1}{\partial y} v_1 + \frac{\partial N_2}{\partial y} v_2 + \frac{\partial N_3}{\partial y} \cdot v_3$$

$$\gamma_{xy} = \frac{\partial N_1}{\partial y} u_1 + \frac{\partial N_2}{\partial y} u_2 + \frac{\partial N_3}{\partial y} u_3 + \frac{\partial N_1}{\partial x} v_1 + \frac{\partial N_2}{\partial x} v_2 + \frac{\partial N_3}{\partial x} v_3$$

→ (a)

Arranging the strain e_x, e_y, γ_{xy} in matrix form

$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

→ (b)

$$N_1 = \frac{p_1 + q_1 x + r_1 y}{2A} \Rightarrow \frac{\partial N_1}{\partial x} = \frac{q_1}{2A} \qquad \frac{\partial N_1}{\partial y} = \frac{r_1}{2A}$$

$$N_2 = \frac{p_2 + q_2 x + r_2 y}{2A} \Rightarrow \partial N_2 = \frac{q_2}{2A} \qquad \frac{\partial N_2}{\partial y} = \frac{r_2}{2A}$$

$$N_3 = \frac{P_3 + q_3 x + r_3 y}{2A} \Rightarrow \frac{\partial N_3}{\partial x} = \frac{q_3}{2A}, \quad \frac{\partial N_3}{\partial y} = \frac{r_3}{2A} \rightarrow (c)$$

Subs Eq. (c) in (b)

we get

$$\begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \rightarrow (d)$$

Which is of the form

$$\{e\} = [B]\{u\}.$$

Sol

$$\boxed{[B] = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}}$$

When

$$q_1 = y_2 - y_3 \qquad r_1 = x_3 - x_2$$

$$q_2 = y_3 - y_1 \qquad r_2 = x_1 - x_3$$

$$q_3 = y_1 - y_2 \qquad r_3 = x_2 - x_1$$

Stiffness Matrix Equation for 2D Element (CST.)

We know $[K] = \int_V [B]^T [D][B]\, dV$

$[K] = [B]^T \cdot [D][B] \cdot V = [B]^T [D][B] = A \cdot t$

$$\boxed{[K] = [B]^T [D][B] \cdot A \cdot t}$$

Area of tri el $= \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$t \rightarrow$ thickness of element

$[B] \rightarrow$ Strain Displacement mat.

$$[B] = \frac{1}{2A} \begin{bmatrix} q_1 & 0 & q_2 & 0 & q_3 & 0 \\ 0 & r_1 & 0 & r_2 & 0 & r_3 \\ r_1 & q_1 & r_2 & q_2 & r_3 & q_3 \end{bmatrix}$$

$[D] \rightarrow$ Stress Strain relationship matrix

$$[D] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$ (Plane Stress prob)

$$[D] = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} (1-\mu) & \mu & 0 \\ \mu & (1-\mu) & 0 \\ 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}$$

(Plain Strain prob)

~~E~~ $E \rightarrow$ Young mod

$\mu \rightarrow$ Poisson Ratio.